

EXTENDING STRUCTURES I: THE LEVEL OF GROUPS

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ABSTRACT. Let H be a group and E a set such that $H \subseteq E$. We shall describe and classify up to an isomorphism of groups that stabilizes H the set of all group structures \cdot that can be defined on E such that H is a subgroup of (E, \cdot) . Thus we solve at the level of groups what we have called the *extending structures problem*. A general product, which we call the unified product, is constructed such that both the crossed product and the Takeuchi's bicrossed product of two groups are special cases of it. It is associated to H and to a system $((S, 1_S, *), \triangleleft, \triangleright, f)$ called a group extending structure and we denote it by $H \ltimes S$. There exists a group structure \cdot on E containing H as a subgroup if and only if there exists an isomorphism of groups $(E, \cdot) \cong H \ltimes S$, for some group extending structure $((S, 1_S, *), \triangleleft, \triangleright, f)$. All such group structures \cdot on E are classified up to an isomorphism of groups that stabilizes H by a cohomological type set $\mathcal{K}_{\ltimes}^2(H, (S, 1_S))$. A general Schreier theorem is proved and an answer to a question of Kuperberg is given, both being special cases of our classification result. The above construction is related to the existence of hidden symmetries of H -principal bundles at the level of 0-dimensional manifolds (discrete sets).

INTRODUCTION

The present paper is the starting point for a study concerning what we have called the *extending structures problem* or the ES-problem for short. The ES-problem can be formulated at the most general level using category theory language [3]. At the level of groups, i.e. corresponding to the forgetful functor $F : \mathcal{Gr} \rightarrow \mathcal{Set}$ from the category of groups to the category of sets, the ES problem has a very tempting statement:

(Gr) Extending structures problem. *Let H be a group and E a set such that $H \subseteq E$. Describe and classify up to an isomorphism that stabilizes H the set of all group structures \cdot that can be defined on E such that H is a subgroup of (E, \cdot) .*

In other words, the (Gr) ES-problem is trying to provide an answer to the very natural question: to what extent a group structure on H can be extended beyond H to a bigger set which contains H as a subset in such a way that H would become a subgroup within the new structure. As we showed in [3], this problem can be stated in many other fields of study such as: Lie groups or Lie algebras, algebras or coalgebras, Hopf algebras and quantum groups, locally compact groups or locally compact quantum groups etc. The (Gr) ES-problem generalizes and unifies two famous problems in the theory of groups

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which served as models for our construction: the *extension problem* of Hölder [12] and the *factorization problem* of Ore [18]. Let us explain this briefly. Consider two groups H and G . The extension problem of Hölder consists of describing and classifying all groups E containing H as a normal subgroup such that $E/H \cong G$. An important step related to the extension problem was made by Schreier: the crossed product associated to a crossed system (H, G, α, f) of groups was constructed and it was proven that any extension E of H by G is equivalent to a crossed product extension (Remark 1.1). The extension problem was one of the most studied problems in group theory in the last century and it has been the starting point of new subjects in mathematics such as cohomology of groups, homological algebra, crossed products of groups acting on algebras, crossed products of Hopf algebras acting on algebras, crossed products for von Neumann algebras etc. One of the most important contributions to the extension problem was given by S. Eilenberg and S. MacLane in two fundamental papers [10]. For more details and references on the extension problem we refer to the monograph [1].

The factorization problem is a "dual" of the extension problem and it was formulated by Ore [18] but its roots descend to E. Maillet's paper [17]. It consists of describing and classifying up to an isomorphism all groups E that factorize through H and G : i.e. E contains H and G as subgroups such that $E = HG$ and $H \cap G = \{1\}$. The dual version of Schreier's theorem was proven by Takeuchi [21]: the bicrossed product associated to a matched pair of groups $(H, G, \triangleleft, \triangleright)$ was constructed and it was proven that a group E factorizes through H and G if and only if E is isomorphic to a bicrossed product $H \bowtie G$. The factorization problem is even more difficult than the more popular extension problem and little progress has been made since then. For instance, in the case of two cyclic groups H and G , not both finite, the problem was started by L. Rédei in [19] and finished by P.M. Cohn in [7]. If H and G are both finite cyclic groups the problem seems to be still open, even though J. Douglas [9] has devoted four papers to the subject. The case of two cyclic groups, one of them being of prime order, was solved recently in [2]. In [8] all bicrossed products $A_r \bowtie S_n$, between an alternating group and a symmetric group are completely described. Ito's theorem [13] remains one of the most famous results about the factorization problem: any product $E = HG$ of two abelian subgroups is a metabelian group. Takeuchi's bicrossed product construction served as a model for similar constructions in other fields of mathematics such as: algebras, coalgebras, groupoids, Hopf algebras, locally compact groups, locally compact quantum groups, Lie algebras, Lie groups. We would like to mention that the bicrossed product, also known as knit product or Zappa-Szep product in the theory of groups, appeared for the first time in group theory in a paper by Zappa [23] and it was rediscovered later on by Szep [22]. For more details and results obtained on the factorization problem and a detailed list of references we refer to the monograph [6].

In the construction of a crossed product associated to a crossed system a weak action $\alpha : G \rightarrow \text{Aut}(H)$ and an α -cocycle $f : G \times G \rightarrow H$ are used, while the construction of a bicrossed product involves two compatible actions $\triangleright : G \times H \rightarrow H$ and $\triangleleft : G \times H \rightarrow G$. Even if their starting points are different, the two constructions have something in common: the crossed product structure as well as the bicrossed product structure are defined on the same set, namely $H \times G$. Moreover, $H \cong (H, 1)$ is a subgroup in both the

crossed as well as the bicrossed product (in fact $H \cong (H, 1)$ is even a normal subgroup in the crossed product). Now, relying on Lagrange's theorem, the (Gr) ES-problem can be restated as follows: let H be a group and $(S, 1_S)$ a pointed set. Which are the group structures \cdot that can be defined on the set $H \times S$ such that $H \cong (H, 1)$ is a subgroup in $(H \times S, \cdot)$. If we require further that $H \cong (H, 1)$ is a *normal* subgroup in $E = H \times S$, then Schreier's classical method would work as follows: let $G := E/H$ be the quotient group, $\pi : E \rightarrow G$ be the canonical projection and $\chi : G \rightarrow E$ be a section of π as a map with $\chi(1_G) = 1_E$. Using χ we can construct a crossed system (H, G, α, f) (also called 'factor sets' in group theory) such that there exists an isomorphism of groups $E \cong H \#_\alpha^f G$, where $H \#_\alpha^f G$ is the crossed product associated to the crossed system (see Remark 1.1 for details). In other words, the group structure of E can be reconstructed from a normal subgroup and the corresponding quotient group. However, if we drop the normality assumption on H (as for example simple groups do not contain non-trivial normal subgroups) the construction can not be performed anymore and we have to come up with a new method of reconstructing a group E from a given subgroup and another set of data. This is what we do in Theorem 2.1, which is the first important result of the paper. Let $H \leq E$ be a subgroup of a group E . Using the axiom of choice, we can pick a retraction $p : E \rightarrow H$ of the canonical inclusion $i : H \hookrightarrow E$ which is left H -linear' in the sense that $p(hx) = hp(x)$ for all $h \in H$ and $x \in E$. Having this application p which is not necessarily a morphism of groups we consider the pointed set $S := p^{-1}(1)$, the fiber of the surjective map p in 1. The group H and the pointed set S are connected by four maps arising from p : two actions $\triangleright = \triangleright_p : S \times H \rightarrow H$, $\triangleleft = \triangleleft_p : S \times H \rightarrow S$, a cocycle $f = f_p : S \times S \rightarrow H$ and a multiplication $* = *_p : S \times S \rightarrow S$ given by:

$$\begin{aligned} s \triangleright h &:= p(sh), & s \triangleleft h &:= p(sh)^{-1}sh \\ f(s_1, s_2) &:= p(s_1 s_2), & s_1 * s_2 &:= p(s_1 s_2)^{-1} s_1 s_2 \end{aligned}$$

for all $s, s_1, s_2 \in S$ and $h \in H$. Using these maps, we shall prove that there exists an isomorphism of groups $E \cong (H \times S, \cdot)$, where the multiplication \cdot on the set $H \times S$ is given by

$$(h_1, s_1) \cdot (h_2, s_2) := \left(h_1(s_1 \triangleright h_2) f(s_1 \triangleleft h_2, s_2), (s_1 \triangleleft h_2) * s_2 \right) \quad (1)$$

for all $h_1, h_2 \in H$ and $s_1, s_2 \in S$. In other words, even if we drop the normality assumption, the group E can still be rebuilt from a subgroup H and the fiber S of an H -linear retraction. Moreover, any group structure \cdot that can be defined on a set E such that a given group H will be contained as a subgroup has the form (1) for some system $(S, \triangleright, \triangleleft, f, *)$. This new type of product will be called *unified product* and it is easily seen that both the crossed and the bicrossed product of groups are special cases of it. Moreover, among the examples of unified products we will consider a new and interesting one called the *twisted product* (Example 2.13). There is, however, a price to pay for this generalization. More precisely, the group G that appears both in the construction of the crossed product as well as the bicrossed product is replaced by a new interesting algebraic object: a pair $(S, *)$ consisting of a set S and a binary operation $* : S \times S \rightarrow S$ that admits a unit, any element is left invertible but is not necessary associative. The associativity condition of $*$ is deformed by the action \triangleleft and the cocycle f , meaning that

the following condition, called the *twisted associativity condition*, is satisfied:

$$(s_1 * s_2) * s_3 = (s_1 \triangleleft f(s_2, s_3)) * (s_2 * s_3) \quad (2)$$

for all $s_1, s_2, s_3 \in S$.

In the next step we will perform the abstract construction of the unified product $H \ltimes S$: it is associated to a group H and a system of data $\Omega(H) = ((S, 1_S, *), \triangleleft, \triangleright, f)$ called extending datum of H . Theorem 2.6 establishes the system of axioms that has to be satisfied by $\Omega(H)$ such that $H \ltimes S$ with the multiplication defined by (1) becomes a group structure, i.e. it is a unified product. In this case $\Omega(H) = ((S, 1_S, *), \triangleleft, \triangleright, f)$ will be called a *group extending structure* of H . Both the crossed system and the matched pair of two groups are special cases of the concept of group extending structure.

Based on Theorem 2.1 and Theorem 2.6 we answer the first part of the (Gr) ES-problem in Corollary 2.8: there exists a group structure \cdot on E such that H is a subgroup of (E, \cdot) if and only if (E, \cdot) is isomorphic to a unified product $H \ltimes S$, for some group extending structure $\Omega(H) = ((S, 1_S, *), \triangleleft, \triangleright, f)$ of H on a pointed set $(S, 1_S)$ such that $|S||H| = |E|$. Theorem 2.15 shows the universality of the construction: the unified product is at the same time an initial object in a certain category and a final object in another category, which is not the dual of the first one.

In the last section we shall answer the classification part of the (Gr) ES-problem. Theorem 2.16 and then Corollary 2.18 give the classification of unified products. For a group H and a set E such that $H \subseteq E$, the set of all group structures \cdot that can be defined on E such that $H \leq (E, \cdot)$ are classified up to an isomorphism of groups $\psi : (E, \cdot) \rightarrow (E, \cdot')$ that stabilizes H , i.e. the following diagram

$$\begin{array}{ccc} H & \xrightarrow{i} & (E, \cdot) \\ \downarrow \text{Id}_H & & \downarrow \psi \\ H & \xrightarrow{i} & (E, \cdot') \end{array} \quad (3)$$

is commutative, where $i : H \hookrightarrow E$ is the canonical inclusion. All such group structures \cdot on E are classified by a cohomological type set $\mathcal{K}_{\ltimes}^2(H, (S, 1_S))$. As a special case, a more restrictive version of the classification is given in Corollary 2.22, which is a general Schreier theorem for unified products. This time all unified products $H \ltimes S$ are classified up to an isomorphism of groups that stabilizes both H and S by a set $\mathcal{H}_{\ltimes}^2(H, (S, 1_S), \triangleleft)$ which plays for the (Gr) ES-problem the same role as the second cohomology group from Hölder's extension problem. Moreover, Corollary 2.23 and Corollary 2.25 give necessary and sufficient conditions for a unified product $H \ltimes S$ to be isomorphic to a crossed product $H \#_{\alpha}^f G$ and respectively to a bicrossed product $H \bowtie G$ such that the isomorphism stabilizes H . Finally, in Corollary 2.26 we give necessary and sufficient conditions for a group E to admit two exact factorizations $E = HS$ and $E = HT$ through two subgroups S and T , answering a recent question of Kuperberg.

Finally, we present two directions for further research in which the above results can be used. One of them is to generalize this unified construction to other fields of study in which the crossed product as well as the bicrossed product are already constructed.

After this paper was written, we made the first step at the level of quantum groups in [3]. The second direction for further study is related to the existence of hidden symmetries of an H -principal bundle in differential geometry. The unified product construction gives an answer in the particular case at the level of 0-dimensional manifolds (discrete sets).

1. PRELIMINARIES

Let $(S, 1_S)$ be a pointed set, i.e. S is a non-empty set and $1_S \in S$ is a fixed element in S . The group structures on a set H will be denoted using multiplicative notation and the unit element will be denoted by 1_H or only 1 when there is no danger of confusion. $\text{Aut}(H)$ denotes the group of automorphisms of a group H and $|S|$ the cardinal of a set S . A map $r : S \rightarrow H$ is called unitary if $r(1_S) = 1_H$. S is called a right H -set if there exists a right action $\triangleleft : S \times H \rightarrow S$ of H on S , i.e.

$$s \triangleleft (h_1 h_2) = (s \triangleleft h_1) \triangleleft h_2 \quad \text{and} \quad s \triangleleft 1_H = s \quad (4)$$

for all $s \in S, h_1, h_2 \in H$. The action $\triangleleft : S \times H \rightarrow S$ is called the trivial action if $s \triangleleft h = s$, for all $s \in S$ and $h \in H$. Similarly the maps $\triangleright : S \times H \rightarrow H$ and $f : S \times S \rightarrow H$ are called trivial maps if $s \triangleright h = h$ and respectively $f(s_1, s_2) = 1_H$, for all $s, s_1, s_2 \in S$ and $h \in H$. If G is a group and $\alpha : G \rightarrow \text{Aut}(H)$ is a map we use the similar notation $\alpha(g)(h) = g \triangleright h$, for all $g \in G$ and $h \in H$.

Crossed product of groups. Let H and G be two groups. We recall the classical construction of a crossed product of groups arising from the extension problem. We adopt the notations and terminology from the survey paper [5]. A *crossed system* of groups is a quadruple (H, G, α, f) , where H and G are two groups, $\alpha : G \rightarrow \text{Aut}(H)$ and $f : G \times G \rightarrow H$ are two maps such that the following compatibility conditions hold:

$$g_1 \triangleright (g_2 \triangleright h) = f(g_1, g_2) ((g_1 g_2) \triangleright h) f(g_1, g_2)^{-1} \quad (5)$$

$$f(g_1, g_2) f(g_1 g_2, g_3) = (g_1 \triangleright f(g_2, g_3)) f(g_1, g_2 g_3) \quad (6)$$

for all $g_1, g_2, g_3 \in G$ and $h \in H$. The crossed system $\Gamma = (H, G, \alpha, f)$ is called *normalized* if $f(1, 1) = 1$. The map $\alpha : G \rightarrow \text{Aut}(H)$ is called a *weak action* and $f : G \times G \rightarrow H$ is called an α -*cocycle*. Let $H \#_\alpha^f G := H \times G$ as a set with a binary operation defined by the formula:

$$(h_1, g_1) \cdot (h_2, g_2) := (h_1(g_1 \triangleright h_2) f(g_1, g_2), g_1 g_2) \quad (7)$$

for all $h_1, h_2 \in H, g_1, g_2 \in G$. It is well known that the multiplication on $H \#_\alpha^f G$ given by (7) is associative if and only if (H, G, α, f) is a crossed system (see for instance [5, Theorem 2.3]). In this case $(H \#_\alpha^f G, \cdot)$ is a group with the unit $1_{H \#_\alpha^f G} = (f(1, 1)^{-1}, 1)$ called the *crossed product of H and G* associated to the crossed system (H, G, α, f) . A crossed product with f the trivial cocycle (that is $f(g_1, g_2) = 1_H$, for all $g_1, g_2 \in G$) is just $H \rtimes_\alpha G$, the semidirect product of H and G .

Remark 1.1. The theorem of Schreier states that any extension E of H by a group G is equivalent to a crossed product extension. We recall in detail this construction for further use. Let $H \trianglelefteq E$ be a normal subgroup of a group E and $G := E/H$ the quotient group. Let $\pi : E \rightarrow G$ be the canonical projection and, using the axiom of choice, let

$\chi : G \rightarrow E$ be a section as a map of π (that is $\pi \circ \chi = \text{Id}_G$) with $\chi(1_G) = 1_E$. Using χ we define a weak action α and a cocycle f by the formulas:

$$\alpha : G \rightarrow \text{Aut}(H), \quad \alpha(g)(h) = g \triangleright h := \chi(g)h\chi(g)^{-1} \quad (8)$$

$$f : G \times G \rightarrow H, \quad f(g_1, g_2) := \chi(g_1)\chi(g_2)\chi(g_1g_2)^{-1} \quad (9)$$

for all $g \in G$, $g_1, g_2 \in G$ and $h \in H$. Then (H, G, α, f) is a normalized crossed system of groups and

$$\theta : H \#_\alpha^f G \rightarrow E, \quad \theta(h, g) := h\chi(g)$$

is an isomorphism of groups. For more details we refer the reader to [5, Theorem 2.6]. In other words any group E can be rebuilt from a normal subgroup and the corresponding factor group as a crossed product.

A reinterpretation of the classical construction from Remark 1.1 is the following Corollary which gives an answer of (Gr) ES-problem in the particular case when we are looking for group structures of E such that H is a normal subgroup of it:

Corollary 1.2. *Let H be a group, E be a set such that $H \subseteq E$. Then any group structure \cdot that can be defined on the set E such that H is a normal subgroup of (E, \cdot) is isomorphic to a crossed product associated to a crossed system $(H, G := E/H, \triangleright, \triangleleft, f)$.*

Bicrossed product of groups. A *matched pair* of groups is a quadruple $(H, G, \triangleright, \triangleleft)$, where $\triangleright : G \times H \rightarrow H$ is a left action of the group G on the set H , $\triangleleft : G \times H \rightarrow G$ is a right action of the group H on the set G and the following two compatibility conditions hold:

$$g \triangleright (h_1 h_2) = (g \triangleright h_1)((g \triangleleft h_1) \triangleright h_2) \quad (10)$$

$$(g_1 g_2) \triangleleft h = (g_1 \triangleleft (g_2 \triangleright h))(g_2 \triangleleft h) \quad (11)$$

for all $h, h_1, h_2 \in H$ and $g, g_1, g_2 \in G$ (see [21]). Let $H \bowtie G := H \times G$ as a set with a binary operation defined by the formula:

$$(h_1, g_1) \cdot (h_2, g_2) = (h_1(g_1 \triangleright h_2), (g_1 \triangleleft h_2)g_2) \quad (12)$$

for all $h_1, h_2 \in H$ and $g_1, g_2 \in G$. It can be easily shown that $H \bowtie G$ is a group with $(1_H, 1_G)$ as a unit if and only if $(H, G, \triangleright, \triangleleft)$ is a matched pair of groups. In this case $H \bowtie G$ is called the *bicrossed product*, *doublecross product*, *knit product* or *Zappa-Szep product* associated to the matched pair $(H, G, \triangleright, \triangleleft)$. Takeuchi proved [21] that a group E factorizes through two subgroups H and G if and only if E is isomorphic to a bicrossed product $H \bowtie G$ associated to a matched pair $(H, G, \triangleright, \triangleleft)$.

2. GROUP EXTENDING STRUCTURE AND UNIFIED PRODUCT

The reconstruction of a group from a subgroup. The abstract definition of the unified product of groups will arise from the following elementary question subsequent to the (Gr) ES-problem:

Let $H \leq E$ be a subgroup in E that is not necessary normal. Can we reconstruct the group structure on E from the one of H and some extra set of datum?

In order to give an answer to the question above we should first note that Schreier's classical construction from Remark 1.1 can not be used anymore. Thus we should come up with a new method of reconstruction. The next theorem indicates the way we can perform this reconstruction. Moreover, it indicates a way to generalize our construction to other mathematical objects like: Hopf algebras, Lie algebras, Lie groups, compact quantum groups, etc.

Theorem 2.1. *Let $H \leq E$ be a subgroup of a group E . Then:*

- (1) *There exists a map $p : E \rightarrow H$ such that $p(1) = 1$ and*

$$p(hx) = hp(x) \quad (13)$$

for all $h \in H$ and $x \in E$.

- (2) *For such a map $p : E \rightarrow H$ we define $S = S_p := p^{-1}(1) = \{x \in E \mid p(x) = 1\}$. Then the multiplication map*

$$\varphi : H \times S \rightarrow E, \quad \varphi(h, s) := hs \quad (14)$$

for all $h \in H$ and $s \in S$ is bijective with the inverse given by

$$\varphi^{-1} : E \rightarrow H \times S, \quad \varphi^{-1}(x) = (p(x), p(x)^{-1}x)$$

for all $x \in E$.

- (3) *For p and S as above there exist four maps $\triangleright = \triangleright_p : S \times H \rightarrow H$, $\triangleleft = \triangleleft_p : S \times H \rightarrow S$, $f = f_p : S \times S \rightarrow H$ and $*$ $= * _p : S \times S \rightarrow S$ given by the formulas*

$$\begin{aligned} s \triangleright h &:= p(sh), & s \triangleleft h &:= p(sh)^{-1}sh \\ f(s_1, s_2) &:= p(s_1s_2), & s_1 * s_2 &:= p(s_1s_2)^{-1}s_1s_2 \end{aligned}$$

for all $s, s_1, s_2 \in S$ and $h \in H$. Using these maps, the unique group structure ' \cdot ' on the set $H \times S$ such that $\varphi : (H \times S, \cdot) \rightarrow E$ is an isomorphism of groups is given by:

$$(h_1, s_1) \cdot (h_2, s_2) := \left(h_1(s_1 \triangleright h_2)f(s_1 \triangleleft h_2, s_2), (s_1 \triangleleft h_2) * s_2 \right) \quad (15)$$

for all $h_1, h_2 \in H$ and $s_1, s_2 \in S$.

Proof. (1) Using the axiom of choice we can fix $\Gamma = (x_i)_{i \in I} \subset E$ to be a system of representatives for the right congruence modulo H in E such that $1 \in \Gamma$. Then for any $x \in E$ there exists a unique $h_x \in H$ and a unique $x_{i_0} \in \Gamma$ such that $x = h_x x_{i_0}$. Thus, there exists a well defined map $p : E \rightarrow H$ given by the formula $p(x) := h_x$, for all $x \in E$. As $1 \in \Gamma$ we have that $p(1) = 1$. Moreover, for any $h \in H$ and $x \in E$ we have that $hx = hh_x x_{i_0}$. Thus $p(hx) = hh_x = hp(x)$, as needed.

(2) We note that $p(x)^{-1}x \in S$ as $p(p(x)^{-1}x) = p(x)^{-1}p(x) = 1$, for all $x \in E$. The rest is straightforward.

(3) First we note that \triangleleft and $*$ are well defined maps. Indeed, using (13), for any $s \in S$ and $h \in H$ we have: $p(s \triangleleft h) = p(p(sh)^{-1}sh) = p(sh)^{-1}p(sh) = 1$, i.e. $s \triangleleft h \in S$. In

the same way $p(s_1 * s_2) = 1$, for any $s_1, s_2 \in S$. Now, we shall prove the following two formulas:

$$p(s_1 h_2 s_2) = (s_1 \triangleright h_2) f(s_1 \triangleleft h_2, s_2) \quad (16)$$

and

$$(s_1 \triangleleft h_2) * s_2 = p(s_1 h_2 s_2)^{-1} s_1 h_2 s_2 \quad (17)$$

for all $s_1, s_2 \in S$ and $h_2 \in H$. Indeed,

$$\begin{aligned} (s_1 \triangleright h_2) f(s_1 \triangleleft h_2, s_2) &= p(s_1 h_2) p((s_1 \triangleleft h_2) s_2) \\ &= p(s_1 h_2) p(p(s_1 h_2)^{-1} s_1 h_2 s_2) \\ &\stackrel{(13)}{=} p(s_1 h_2 s_2) \end{aligned}$$

and

$$\begin{aligned} (s_1 \triangleleft h_2) * s_2 &= (p(s_1 h_2)^{-1} s_1 h_2) * s_2 \\ &= p\left(p(s_1 h_2)^{-1} s_1 h_2 s_2\right)^{-1} p(s_1 h_2)^{-1} s_1 h_2 s_2 \\ &\stackrel{(13)}{=} [p(s_1 h_2)^{-1} p(s_1 h_2 s_2)]^{-1} p(s_1 h_2)^{-1} s_1 h_2 s_2 \\ &= p(s_1 h_2 s_2)^{-1} s_1 h_2 s_2 \end{aligned}$$

as needed. Now, $\varphi : H \times S \rightarrow E$ is a bijection between the set $H \times S$ and the group E . Thus, there exists a unique group structure \cdot on the set $H \times S$ such that φ is an isomorphism of groups. This group structure is obtained by transferring the group structure from E via the bijection φ , i.e. is given by:

$$\begin{aligned} (h_1, s_1) \cdot (h_2, s_2) &= \varphi^{-1}(\varphi(h_1, s_1) \varphi(h_2, s_2)) = \varphi^{-1}(h_1 s_1 h_2 s_2) \\ &= (p(h_1 s_1 h_2 s_2), p(h_1 s_1 h_2 s_2)^{-1} h_1 s_1 h_2 s_2) \\ &\stackrel{(13)}{=} (h_1 p(s_1 h_2 s_2), p(s_1 h_2 s_2)^{-1} s_1 h_2 s_2) \\ &\stackrel{(16),(17)}{=} \left(h_1 (s_1 \triangleright h_2) f(s_1 \triangleleft h_2, s_2), (s_1 \triangleleft h_2) * s_2 \right) \end{aligned}$$

for all $h_1, h_2 \in H$ and $s_1, s_2 \in S$. □

Remarks 2.2. 1. Theorem 2.1 contains as a special case the fact that split monomorphism in the category of groups are described by semidirect products of groups.

Indeed, let $H \leq E$ be a subgroup of E such that the inclusion $i : H \rightarrow E$ is a split monomorphism of groups, that is there exists $p : E \rightarrow H$ a morphism of groups such that $p(h) = h$, for all $h \in H$. Any such morphism p satisfies the condition (13) and hence $S := \text{Ker}(p)$ is a normal subgroup of E . Moreover, the maps constructed in (3) of Theorem 2.1 are the following: \triangleright and f are the trivial maps, $*$ is exactly the subgroup structure of S in E and \triangleleft is the action by conjugation, i.e. $s \triangleleft h = h^{-1} s h$. It follows from (15) that the multiplication on $H \times S$ is exactly the one of the semidirect product $H \rtimes S$, in the right convention, namely:

$$(h_1, s_1) \cdot (h_2, s_2) = \left(h_1 h_2, (s_1 \triangleleft h_2) s_2 \right)$$

for all $h_1, h_2 \in H$ and $s_1, s_2 \in S$. Thus $\varphi : H \rtimes S \rightarrow E$ is an isomorphism of groups.

2. As $1 \in S$ and $p(s) = 1$, for all $s \in S$ the maps $\triangleright = \triangleright_p$, $\triangleleft = \triangleleft_p$, $f = f_p$ and $*$ $=$ $*_p$ constructed in (3) of Theorem 2.1 satisfy the following normalization conditions:

$$s \triangleright 1 = 1, \quad 1 \triangleright h = h, \quad 1 \triangleleft h = 1, \quad s \triangleleft 1 = s \quad (18)$$

$$f(s, 1) = f(1, s) = 1, \quad s * 1 = 1 * s = s \quad (19)$$

for all $s \in S$ and $h \in H$. Hence, the multiplication $*$ on S has a unit but is not necessary associative. In fact, we can easily prove that it satisfies the following compatibility:

$$(s_1 * s_2) * s_3 = (s_1 \triangleleft f(s_2, s_3)) * (s_2 * s_3) \quad (20)$$

for all $s_1, s_2, s_3 \in S$, i.e. $*$ is associative up to the pair (\triangleleft, f) . Moreover, any element $s \in S$ is left invertible in $(S, *)$; more precisely we can show that for any $s \in S$ there exists a unique element $s' \in S$ such that $s' * s = 1$.

Indeed, the multiplication given by (15) is a group structure on $H \times S$ having $(1, 1)$ as a unit. In particular, for any $s \in S$ there exists a unique $h' \in H$ and $s' \in S$ such that $(h', s') \cdot (1, s) = (1, 1)$. Thus, taking into account the normalizing conditions (18) we obtain that $(h'f(s', s), s' * s) = (1, 1)$, i.e. in particular it follows that $s' * s = 1$.

The abstract construction of the unified product. Let H be a group and E a set such that $H \subseteq E$. Theorem 2.1 describes the way any group structure \cdot on the set E such that H is a subgroup of (E, \cdot) should look like. It remains to show what type of abstract axioms should the system of maps $(*, \triangleleft, \triangleright, f)$ satisfy such that (15) is indeed a group structure. This will be done bellow.

Definition 2.3. Let H be a group. An *extending datum* of H is a system $\Omega(H) = ((S, 1_S, *), \triangleleft, \triangleright, f)$ where:

(1) $(S, 1_S)$ is a pointed set, $*$: $S \times S \rightarrow S$ is a binary operation such that for any $s \in S$

$$s * 1_S = 1_S * s = s \quad (21)$$

(2) The maps $\triangleleft : S \times H \rightarrow S$, $\triangleright : S \times H \rightarrow H$ and $f : S \times S \rightarrow H$ satisfy the following normalization conditions for any $s \in S$ and $h \in H$:

$$s \triangleleft 1_H = s, \quad 1_S \triangleleft h = 1_S, \quad 1_S \triangleright h = h, \quad s \triangleright 1_H = 1_H, \quad f(s, 1_S) = f(1_S, s) = 1_H \quad (22)$$

Let H be a group and $\Omega(H) = ((S, 1_S, *), \triangleleft, \triangleright, f)$ an extending datum of H . We denote by $H \ltimes_{\Omega(H)} S := H \ltimes S$ the set $H \times S$ with the binary operation defined by the formula:

$$(h_1, s_1) \cdot (h_2, s_2) := \left(h_1(s_1 \triangleright h_2)f(s_1 \triangleleft h_2, s_2), (s_1 \triangleleft h_2) * s_2 \right) \quad (23)$$

for all $h_1, h_2 \in H$ and $s_1, s_2 \in S$.

Definition 2.4. Let H be a group and $\Omega(H) = ((S, 1_S, *), \triangleleft, \triangleright, f)$ an extending datum of H . The object $H \ltimes S$ introduced above is called *the unified product of H and $\Omega(H)$* if $H \ltimes S$ is a group with the multiplication given by (23). In this case the extending datum $\Omega(H)$ is called a *group extending structure* of H . The maps \triangleright and \triangleleft are called the *actions* of $\Omega(H)$ and f is called the $(\triangleright, \triangleleft)$ -*cocycle* of $\Omega(H)$.

The multiplication formula defined by (23) arise naturally from (3) of Theorem 2.1 which was our starting point on solving the extending structures problem for groups.

Remark 2.5. Using the normalizing conditions (21) and (22) it is straightforward to prove that $(1_H, 1_S)$ is a unit of the multiplication (23) and the following cross relations hold in $H \ltimes S$:

$$(h_1, 1_S) \cdot (h_2, s_2) = (h_1 h_2, s_2) \quad (24)$$

$$(h_1, s_1) \cdot (1_H, s_2) = (h_1 f(s_1, s_2), s_1 * s_2) \quad (25)$$

$$(h_1, s_1) \cdot (h_2, 1_S) = (h_1 (s_1 \triangleright h_2), s_1 \triangleleft h_2) \quad (26)$$

for all $h_1, h_2 \in H$ and $s_1, s_2 \in S$.

Next, we indicate the abstract system of axioms that need to be satisfied by the functions $(*, \triangleleft, \triangleright, f)$ such that $H \ltimes S$ becomes a unified product.

Theorem 2.6. *Let H be a group and $\Omega(H) = ((S, 1_S, *), \triangleleft, \triangleright, f)$ an extending datum of H . The following statements are equivalent:*

- (1) *$A \ltimes H$ is an unified product;*
- (2) *The following compatibilities hold for any $s, s_1, s_2, s_3 \in S$ and $h, h_1, h_2 \in H$:*
 - (ES1) *The map $\triangleleft : S \times H \rightarrow S$ is a right action of the group H on the set S ;*
 - (ES2) $(s_1 * s_2) * s_3 = (s_1 \triangleleft f(s_2, s_3)) * (s_2 * s_3)$
 - (ES3) $s \triangleright (h_1 h_2) = (s \triangleright h_1) ((s \triangleleft h_1) \triangleright h_2)$
 - (ES4) $(s_1 * s_2) \triangleleft h = (s_1 \triangleleft (s_2 \triangleright h)) * (s_2 \triangleleft h)$
 - (ES5) $(s_1 \triangleright (s_2 \triangleright h)) f(s_1 \triangleleft (s_2 \triangleright h), s_2 \triangleleft h) = f(s_1, s_2) ((s_1 * s_2) \triangleright h)$
 - (ES6) $f(s_1, s_2) f(s_1 * s_2, s_3) = (s_1 \triangleright f(s_2, s_3)) f(s_1 \triangleleft f(s_2, s_3), s_2 * s_3)$
 - (ES7) *For any $s \in S$ there exists $s' \in S$ such that $s' * s = 1_S$.*

Before going into the proof of the theorem we note that (ES3) and (ES4) are exactly, mutatis-mutandis, the compatibility conditions (10) and (11) from the definition of a matched pair of groups while (ES5) and (ES6) are deformations via the right action \triangleleft of the compatibility conditions (5) and (6) from the definition of a crossed system of groups. The axiom (ES1) is called the *twisted associativity condition* as it measures how far $*$ is from being associative, i.e. from being a group structure on S .

Proof. We know that $(1_H, 1_S)$ is a unit for the operation (23). We prove now that the operation \cdot given by (23) is associative if and only if the compatibility conditions (ES1) – (ES6) hold. Assume first that \cdot is associative and let $h, h_1, h_2 \in H$ and $s, s_1, s_2 \in S$. The associativity condition

$$[(1_H, s) \cdot (h_1, 1_S)] \cdot (h_2, 1_S) = (1_H, s) \cdot [(h_1, 1_S) \cdot (h_2, 1_S)]$$

gives, after we use the cross relations (26) and (24), $(s \triangleright h_1, s \triangleleft h_1) \cdot (h_2, 1_S) = (1_H, s) \cdot (h_1 h_2, 1_S)$. Thus $((s \triangleright h_1) ((s \triangleleft h_1) \triangleright h_2), (s \triangleleft h_1) \triangleleft h_2) = (s \triangleright (h_1 h_2), s \triangleleft (h_1 h_2))$ and hence

(*ES1*) and (*ES3*) hold. Now, we write the associativity condition $[(1_H, s_1) \cdot (1_H, s_2)] \cdot (1_H, s_3) = (1_H, s_1) \cdot [(1_H, s_2) \cdot (1_H, s_3)]$ and compute this equality using the cross relation (25) we obtain the fact that the compatibility conditions (*ES2*) and (*ES6*) hold. Finally, if we write the associativity condition $[(1_H, s_1) \cdot (1_H, s_2)] \cdot (h, 1_S) = (1_H, s_1) \cdot [(1_H, s_2) \cdot (h, 1_S)]$ and use (25) and then (26) we obtain precisely the fact that (*ES4*) and (*ES5*) hold.

Conversely, assume that the compatibility conditions (*ES1*) – (*ES6*) hold. Then for any $h_1, h_2, h_3 \in H$ and $s_1, s_2, s_3 \in S$ we have:

$$\begin{aligned}
& (h_1, s_1) \cdot [(h_2, s_2) \cdot (h_3, s_3)] = \\
& = \left(h_1 [s_1 \triangleright (h_2(s_2 \triangleright h_3)f(s_2 \triangleleft h_3, s_3))] f(s_1 \triangleleft (h_2(s_2 \triangleright h_3)f(s_2 \triangleleft h_3, s_3)), (s_2 \triangleleft h_3) * s_3), \right. \\
& \quad \left. [s_1 \triangleleft (h_2(s_2 \triangleright h_3)f(s_2 \triangleleft h_3, s_3))] * ((s_2 \triangleleft h_3) * s_3) \right) \\
& \stackrel{(ES3)}{=} \left(h_1(s_1 \triangleright h_2)[(s_1 \triangleleft h_2) \triangleright ((s_2 \triangleright h_3)f(s_2 \triangleleft h_3, s_3))] f(s_1 \triangleleft (h_2(s_2 \triangleright h_3)f(s_2 \triangleleft h_3, s_3)), \right. \\
& \quad \left. (s_2 \triangleleft h_3) * s_3), [s_1 \triangleleft (h_2(s_2 \triangleright h_3)f(s_2 \triangleleft h_3, s_3))] * ((s_2 \triangleleft h_3) * s_3) \right) \\
& \stackrel{(ES1)}{=} \left(h_1(s_1 \triangleright h_2)[(s_1 \triangleleft h_2) \triangleright ((s_2 \triangleright h_3)f(s_2 \triangleleft h_3, s_3))] f((s_1 \triangleleft h_2) \triangleleft [(s_2 \triangleright h_3)f(s_2 \triangleleft h_3, s_3)], \right. \\
& \quad \left. (s_2 \triangleleft h_3) * s_3), [s_1 \triangleleft (h_2(s_2 \triangleright h_3)f(s_2 \triangleleft h_3, s_3))] * ((s_2 \triangleleft h_3) * s_3) \right) \\
& \stackrel{(ES3)}{=} \left(h_1(s_1 \triangleright h_2)((s_1 \triangleleft h_2) \triangleright (s_2 \triangleright h_3))[(s_1 \triangleleft h_2) \triangleleft (s_2 \triangleright h_3)] \triangleright f(s_2 \triangleleft h_3, s_3) \right. \\
& \quad \left. f((s_1 \triangleleft h_2) \triangleleft [(s_2 \triangleright h_3)f(s_2 \triangleleft h_3, s_3)], (s_2 \triangleleft h_3) * s_3), [s_1 \triangleleft (h_2(s_2 \triangleright h_3)f(s_2 \triangleleft h_3, s_3))] * \right. \\
& \quad \left. ((s_2 \triangleleft h_3) * s_3) \right) \\
& \stackrel{(ES5)}{=} \left(h_1(s_1 \triangleright h_2)f(s_1 \triangleleft h_2, s_2)[((s_1 \triangleleft h_2) * s_2) \triangleright h_3]f((s_1 \triangleleft h_2) \triangleleft (s_2 \triangleright h_3), s_2 \triangleleft h_3)^{-1} \right. \\
& \quad \left. [((s_1 \triangleleft h_2) \triangleleft (s_2 \triangleright h_3)) \triangleright f(s_2 \triangleleft h_3, s_3)]f((s_1 \triangleleft h_2) \triangleleft [(s_2 \triangleright h_3)f(s_2 \triangleleft h_3, s_3)], \right. \\
& \quad \left. (s_2 \triangleleft h_3) * s_3), [s_1 \triangleleft (h_2(s_2 \triangleright h_3)f(s_2 \triangleleft h_3, s_3))] * ((s_2 \triangleleft h_3) * s_3) \right) \\
& \stackrel{(ES6)}{=} \left(h_1(s_1 \triangleright h_2)f(s_1 \triangleleft h_2, s_2)[((s_1 \triangleleft h_2) * s_2) \triangleright h_3]f((s_1 \triangleleft h_2) \triangleleft (s_2 \triangleright h_3), s_2 \triangleleft h_3)^{-1} \right. \\
& \quad \left. f((s_1 \triangleleft h_2) \triangleleft (s_2 \triangleright h_3), s_2 \triangleleft h_3)f([(s_1 \triangleleft h_2) \triangleleft (s_2 \triangleright h_3)] * (s_2 \triangleleft h_3), s_3), \right. \\
& \quad \left. [s_1 \triangleleft (h_2(s_2 \triangleright h_3)f(s_2 \triangleleft h_3, s_3))] * ((s_2 \triangleleft h_3) * s_3) \right) \\
& = \left(h_1(s_1 \triangleright h_2)f(s_1 \triangleleft h_2, s_2)[((s_1 \triangleleft h_2) * s_2) \triangleright h_3]f([(s_1 \triangleleft h_2) \triangleleft (s_2 \triangleright h_3)] * (s_2 \triangleleft h_3), s_3), \right. \\
& \quad \left. [s_1 \triangleleft (h_2(s_2 \triangleright h_3)f(s_2 \triangleleft h_3, s_3))] * ((s_2 \triangleleft h_3) * s_3) \right) \\
& \stackrel{(ES4)}{=} \left(h_1(s_1 \triangleright h_2)f(s_1 \triangleleft h_2, s_2)[((s_1 \triangleleft h_2) * s_2) \triangleright h_3]f([(s_1 \triangleleft h_2) * s_2] \triangleleft h_3, s_3), \right. \\
& \quad \left. [s_1 \triangleleft (h_2(s_2 \triangleright h_3)f(s_2 \triangleleft h_3, s_3))] * ((s_2 \triangleleft h_3) * s_3) \right)
\end{aligned}$$

$$\begin{aligned}
& \stackrel{(ES1)}{=} \left(h_1(s_1 \triangleright h_2) f(s_1 \triangleleft h_2, s_2) [((s_1 \triangleleft h_2) * s_2) \triangleright h_3] f([(s_1 \triangleleft h_2) * s_2] \triangleleft h_3, s_3), \right. \\
& \quad \left. \underline{[s_1 \triangleleft (h_2(s_2 \triangleright h_3))]} \triangleleft f(s_2 \triangleleft h_3, s_3) * [(s_2 \triangleleft h_3)] * s_3 \right) \\
& \stackrel{(ES2)}{=} \left(h_1(s_1 \triangleright h_2) f(s_1 \triangleleft h_2, s_2) [((s_1 \triangleleft h_2) * s_2) \triangleright h_3] f([(s_1 \triangleleft h_2) * s_2] \triangleleft h_3, s_3), \right. \\
& \quad \left. \underline{[(s_1 \triangleleft (h_2(s_2 \triangleright h_3))) * (s_2 \triangleleft h_3)] * s_3} \right) \\
& \stackrel{(ES1)}{=} \left(h_1(s_1 \triangleright h_2) f(s_1 \triangleleft h_2, s_2) [((s_1 \triangleleft h_2) * s_2) \triangleright h_3] f([(s_1 \triangleleft h_2) * s_2] \triangleleft h_3, s_3), \right. \\
& \quad \left. [((s_1 \triangleleft h_2) \triangleleft (s_2 \triangleright h_3)) * (s_2 \triangleleft h_3)] * s_3 \right) \\
& \stackrel{(ES4)}{=} \left(h_1(s_1 \triangleright h_2) f(s_1 \triangleleft h_2, s_2) [((s_1 \triangleleft h_2) * s_2) \triangleright h_3] f([(s_1 \triangleleft h_2) * s_2] \triangleleft h_3, s_3), \right. \\
& \quad \left. [((s_1 \triangleleft h_2) * s_2) \triangleleft h_3] * s_3 \right) = [(h_1, s_1) \cdot (h_2, s_2)] \cdot (h_3, s_3)
\end{aligned}$$

as needed. To conclude, we proved that $(H \ltimes S, \cdot)$ is a monoid if and only if (ES1)–(ES6) hold.

Assume now that $(H \ltimes S, \cdot)$ is a monoid: it remains to be proved that the monoid is actually a group if and only if (ES7) holds. Indeed, in the monoid $(H \ltimes S, \cdot)$ we have:

$$(h, 1_S) \cdot (1_H, s) = (h, s), \quad (h_1, 1_S) \cdot (h_2, 1_S) = (h_1 h_2, 1_S)$$

for all $h, h_1, h_2 \in H$ and $s \in S$. In particular, any element of the form $(h, 1_S)$, for $h \in H$ is invertible in $(H \ltimes S, \cdot)$. Now, a monoid is a group if and only if each of his elements has a left inverse. As \cdot is associative it follows from:

$$(h^{-1}, 1_S) \cdot (h, s) = (1_H, s)$$

that $(H \ltimes S, \cdot)$ is a group if and only if $(1_H, s)$ has a left inverse for all $s \in S$. Hence, for any $s \in S$ there exist elements $s' \in S$ and $h' \in H$ such that

$$(h', s') \cdot (1_H, s) = (h' f(s', s), s' * s) = (1_H, 1_S)$$

This is of course equivalent to the fact that $s' * s = 1_S$ for all $s \in S$ and $h' = f(s', s)^{-1}$.

The proof is now finished. We note that the inverse of an element (h, s) in the group $(H \ltimes S, \cdot)$ is given by the formula

$$(h, s)^{-1} = (f(s', s)^{-1}, s') \cdot (h^{-1}, 1_S) = (f(s', s)^{-1}(s' \triangleright h^{-1}), s' \triangleleft h^{-1})$$

where $s' * s = 1_S$. □

Remark 2.7. Let H be a group, $\Omega(H) = ((S, 1_S, *), \triangleleft, \triangleright, f)$ a group extending datum of H and $H \ltimes_\Omega S$ the associated unified product. Then the canonical inclusion

$$i_H : H \rightarrow H \ltimes_\Omega S, \quad i_H(h) := (h, 1_S)$$

is a morphism of groups and the map

$$p_H : H \ltimes_\Omega S \rightarrow H, \quad p_H(h, s) := h$$

satisfies the condition (13) of Theorem 2.1. Moreover, if we identify $H \cong (H, 1_S) \leq H \ltimes_\Omega S$ and $S \cong (1_H, S) \subset H \ltimes_\Omega S$ we can easily show that the maps $*$, \triangleleft , \triangleright and f

from the definition of $\Omega(H)$ are exactly the ones given in (3) of Theorem 2.1 associated to the splitting map p_H .

We record these observations as follows:

Corollary 2.8. *Let H be a group and E a set such that $H \subseteq E$. Then there exists a one-to-one correspondence between:*

- (i) *the set of all group structures \cdot on the set E such that H is a subgroup of (E, \cdot) .*
- (ii) *the set of all group extending structures $\Omega(H) = ((S, 1_S, *), \triangleleft, \triangleright, f)$ of H defined on a pointed set $(S, 1_S)$ such that $|S||H| = |E|$.*

Proof. It follows from Remark 2.7, Theorem 2.1 and Theorem 2.6. \square

Examples of unified products. In this subsection we provide some examples and special cases of unified products. First of all it follows from Theorem 2.1 and Remark 2.7 that any group extending structure of a group H is constructed as follows:

Corollary 2.9. *Let H be a group. Then $\Omega(H) = ((S, 1_S, *), \triangleleft, \triangleright, f)$ is a group extending structure of H if and only if there exists a group E containing H as a subgroup and an unitary map $p : E \rightarrow H$ with $p(hx) = hp(x)$, for all $h \in H$ and $x \in E$ such that $S, *, \triangleright, \triangleleft$ and f are given by:*

$$S := p^{-1}(1), \quad s \triangleright h := p(sh), \quad s \triangleleft h := p(sh)^{-1}sh, \quad (27)$$

$$f(s_1, s_2) := p(s_1 s_2), \quad s_1 * s_2 := p(s_1 s_2)^{-1} s_1 s_2 \quad (28)$$

for all $s, s_1, s_2 \in S$ and $h \in H$.

Next we show that the unified product unifies both the crossed product as well as the bicrossed product of two groups.

Example 2.10. Let $\Omega(H) = ((S, 1_S, *), \triangleleft, \triangleright, f)$ be an extending datum of H such that \triangleleft is the trivial action, that is $s \triangleleft h := s$, for all $s \in S$ and $h \in H$. Then $\Omega(H)$ is a group extending structure of H if and only if $(S, *)$ is a group structure on the set S and $(H, (S, *), \triangleright, f)$ is a crossed system of groups. In this case, the associated unified product $H \rtimes_{\Omega} S = H \#_{\triangleright}^f G$ is the crossed product of the two groups.

Furthermore, Corollary 2.23 gives necessary and sufficient conditions for a unified product $H \rtimes_{\Omega} S$ to be isomorphic to a crossed product $H \#_{\triangleright}^f G$ such that the isomorphism stabilizes H .

Example 2.11. Let $\Omega(H) = ((S, 1_S, *), \triangleleft, \triangleright, f)$ be an extending datum of H such that f is the trivial cocycle, that is $f(s_1, s_2) = 1$, for all $s_1, s_2 \in S$. Then $\Omega(H)$ is a group extending structure of H if and only if $(S, *)$ is a group structure on the set S and $(H, (S, *), \triangleright, \triangleleft)$ is a matched pair of groups. In this case, the associated unified product $A \rtimes_{\Omega} H = H \bowtie G$ is the bicrossed product of the two groups.

Corollary 2.25 gives necessary and sufficient conditions for a unified product $H \rtimes_{\Omega} S$ to be isomorphic to a bicrossed product $H \bowtie G$ such that the isomorphism stabilizes H .

Example 2.12. There exist examples of groups that cannot be written either as a crossed product or as a bicrossed product of two groups of smaller order. Such a group should be a simple group (otherwise it can be written as a crossed product). The simple group of smallest order that cannot be written as a bicrossed product is the alternating group A_6 ([24]). The above results allows us to write A_6 , and in fact any other simple group which is not a bicrossed product, as a unified product between one of its subgroups and an extending structure. For instance, we can write

$$A_6 \cong A_4 \ltimes_{\Omega} S$$

for an extending structure $\Omega(A_4) = ((S, 1_S, *), \triangleleft, \triangleright, f)$, where S is a set with 30 elements.

It is also interesting to write down the case where the action \triangleright is trivial as it reveals a new type of product of groups.

Example 2.13. Let $\Omega(H) = ((S, 1_S, *), \triangleleft, \triangleright, f)$ be an extending datum of H such that \triangleright is the trivial action, that is $s \triangleright h := h$ for all $s \in S$ and $h \in H$. Then, $\Omega(H)$ is a group extending structure of H if and only if \triangleleft is a right action of H on S , any element of S is right invertible and the following compatibility conditions hold:

$$\begin{aligned} (s_1 * s_2) * s_3 &= (s_1 \triangleleft f(s_2, s_3)) * (s_2 * s_3) \\ f(s_1 \triangleleft h, s_2 \triangleleft h) &= h^{-1} f(s_1, s_2) h \\ (s_1 * s_2) \triangleleft h &= (s_1 \triangleleft h) * (s_2 \triangleleft h) \\ f(s_1, s_2) f(s_1 * s_2, s_3) &= f(s_2, s_3) f(s_1 \triangleleft f(s_2, s_3), s_2 * s_3) \end{aligned}$$

for all $s, s_1, s_2, s_3 \in S$ and $h, h_1, h_2 \in H$. The associated unified product $A \ltimes_{\Omega} H$ will be denoted by $H \times^{f, \triangleleft} S$. Explicitly, $H \times^{f, \triangleleft} S := H \times S$ as a set with the binary operation defined by the formula:

$$(h_1, s_1) \cdot (h_2, s_2) := (h_1 h_2 f(s_1 \triangleleft h_2, s_2), (s_1 \triangleleft h_2) * s_2) \quad (29)$$

for all $h_1, h_2 \in H$ and $s_1, s_2 \in S$. Then $H \times^{f, \triangleleft} S$ is a group and we call it the *twisted product* of H and $\Omega(H)$.

Finally, an example of a group extending structure was constructed in [4] as follows:

Example 2.14. Let H be a group, $(S, 1_S, *)$ a pointed set with a binary operation $*$: $S \times S \rightarrow S$ having 1_S as a unit. Let $\triangleleft : S \times H \rightarrow S$ be a map such that (S, \triangleleft) is a right H -set and $\gamma : S \rightarrow H$ be a map with $\gamma(1_S) = 1_H$ such that the following compatibilities hold

$$(x * y) * z = \left(x \triangleleft (\gamma(y) \gamma(z) \gamma(y * z)^{-1}) \right) * (y * z) \quad (30)$$

$$(x * y) \triangleleft g = \left(x \triangleleft (\gamma(y) g \gamma(y \triangleleft g)^{-1}) \right) * (y \triangleleft g) \quad (31)$$

for all $g \in H, x, y, z \in S$. Using the transition map γ we define a left action \triangleright and a cocycle f via:

$$x \triangleright g := \gamma(x) g \gamma(x \triangleleft g)^{-1}, \quad f(x, y) := \gamma(x) \gamma(y) \gamma(x * y)^{-1}$$

for all $x, y \in S$ and $g \in H$. Then we can prove that $\Omega(H) = ((S, 1_S, *), \triangleleft, \triangleright, f)$ is a group extending structure of H .

The universal properties of the unified product. In this subsection we prove the universality of the unified product. Let H be a group and $\Omega(H) = ((S, 1_S, *), \triangleleft, \triangleright, f)$ a group extending structure of H . We associate to $\Omega(H)$ two categories ${}_{\Omega(H)}\mathcal{C}$ and $\mathcal{D}_{\Omega(H)}$ such that the unified product becomes an initial object in the first category and a final object in the second category (note that is not the dual of the first category). Define the category ${}_{\Omega(H)}\mathcal{C}$ as follows: the objects of ${}_{\Omega(H)}\mathcal{C}$ are pairs $(G, (u, v))$, where G is a group, $h : H \rightarrow G$ is a morphism of groups and $v : S \rightarrow G$ is a map such that:

$$v(s_1)v(s_2) = u(f(s_1, s_2))v(s_1 * s_2) \quad (32)$$

$$v(s)u(h) = u(s \triangleright h)v(s \triangleleft h) \quad (33)$$

for all $s, s_1, s_2 \in S$ and $h \in H$. The morphisms of the category $f : (G_1, (u_1, v_1)) \rightarrow (G_2, (u_2, v_2))$ are morphisms of groups $f : G_1 \rightarrow G_2$ such that : $f \circ u_1 = u_2$ and $f \circ v_1 = v_2$. Define the category $\mathcal{D}_{\Omega(H)}$ as follows: the objects of $\mathcal{D}_{\Omega(H)}$ are pairs $(G, (u, v))$, where G is a group, $u : G \rightarrow H$, $v : G \rightarrow S$ are maps such that:

$$u(xy) = u(x)[v(x) \triangleright u(y)]f(v(x) \triangleleft u(y), v(y)) \quad (34)$$

$$v(xy) = [v(x) \triangleleft u(y)] * v(y) \quad (35)$$

for all $x, y \in G$ while the morphisms of this category $f : (G_1, (u_1, v_1)) \rightarrow (G_2, (u_2, v_2))$ are morphisms of groups $f : G_1 \rightarrow G_2$ such that: $u_2 \circ f = u_1$ and $v_2 \circ f = v_1$.

Theorem 2.15. *Let H be a group and $\Omega(H) = ((S, 1_S, *), \triangleleft, \triangleright, f)$ a group extending structure of H . Then:*

- (1) $(H \rtimes_{\Omega} S, (i_H, i_S))$ is an initial object of ${}_{\Omega(H)}\mathcal{C}$, where $i_H : H \rightarrow H \rtimes_{\Omega} S$ and $i_S : S \rightarrow H \rtimes_{\Omega} S$ are the canonical inclusions;
- (2) $(H \rtimes_{\Omega} S, (\pi_H, \pi_S))$ is a final object of $\mathcal{D}_{\Omega(H)}$, where $\pi_H : H \rtimes_{\Omega} S \rightarrow H$ and $\pi_S : H \rtimes_{\Omega} S \rightarrow S$ are the canonical projections.

Proof. (1) It is easy to see that $(H \rtimes_{\Omega} S, (i_H, i_S))$ is an object in the category ${}_{\Omega(H)}\mathcal{C}$. Let $(G, (u, v))$ be an object in ${}_{\Omega(H)}\mathcal{C}$. We need to prove that there exists a unique morphism of groups $\phi : H \rtimes_{\Omega} S \rightarrow G$ such that the following diagram commutes:

$$\begin{array}{ccccc} H & \xrightarrow{i_H} & H \rtimes_{\Omega} S & \xleftarrow{i_S} & S \\ & \searrow u & \downarrow \phi & \swarrow v & \\ & & G & & \end{array}$$

Assume first that ϕ satisfies the above condition. We obtain: $\phi((h, s)) = \phi((h, 1_S) \cdot (1_H, s)) = \phi((h, 1_S))\phi((1_H, s)) = (\phi \circ i_H)(h)(\phi \circ i_S)(s) = u(h)v(s)$, for all $h \in H$, $s \in S$ and we proved that ϕ is uniquely determined by u and v . The existence of ϕ can be proved as follows: we define $\phi : H \rtimes_{\Omega} S \rightarrow G$ by $\phi((h, s)) := u(h)v(s)$, for all $h \in H$ and

$s \in S$. It follows that:

$$\begin{aligned}
\phi((h_1, s_1) \cdot (h_2, s_2)) &= \phi(h_1(s_1 \triangleright h_2)f(s_1 \triangleleft h_2, s_2), (s_1 \triangleleft h_2) * s_2) \\
&= u(h_1)u(s_1 \triangleright h_2)u(f(s_1 \triangleleft h_2, s_2))v((s_1 \triangleleft h_2) * s_2) \\
&\stackrel{(32)}{=} u(h_1)u(s_1 \triangleright h_2)v(s_1 \triangleleft h_2)v(s_2) \\
&\stackrel{(33)}{=} u(h_1)v(s_1)u(h_2)v(s_2) \\
&= \phi((h_1, s_1))\phi((h_2, s_2))
\end{aligned}$$

thus ϕ is a morphism of groups. The commutativity of the diagram is obvious.

(2) First we note that $(H \ltimes_{\Omega} S, (\pi_H, \pi_S))$ is an object in the category $\mathcal{D}_{\Omega(H)}$. Let $(G, (u, v))$ be an object in $\mathcal{D}_{\Omega(H)}$. We have to prove that there exists a unique morphism of groups $\psi : G \rightarrow H \ltimes_{\Omega} S$ such that the following diagram commutes:

$$\begin{array}{ccccc}
H & \xleftarrow{\pi_H} & H \ltimes_{\Omega} S & \xrightarrow{\pi_S} & S \\
& \searrow u & \uparrow \psi & \nearrow v & \\
& & G & &
\end{array}$$

Assume first that ψ satisfies the condition above. From the commutativity of the diagram we obtain: $\pi_G \circ \psi = u$ and $\pi_S \circ \psi = v$, i.e. $\psi(g) = (u(g), v(g))$, and we proved that ψ is uniquely determined by u and v . The existence of ψ can be proved as follows: we define $\psi : G \rightarrow H \ltimes_{\Omega} S$ by $\psi(g) := (u(g), v(g))$. It follows from here that:

$$\begin{aligned}
\psi(g_1) \cdot \psi(g_2) &= (u(g_1), v(g_1)) \cdot (u(g_2), v(g_2)) \\
&= (u(g_1)(v(g_1) \triangleright u(g_2))f(v(g_1) \triangleleft u(g_2), v(g_2)), (v(g_1) \triangleleft u(g_2)) * v(g_2)) \\
&\stackrel{(34), (35)}{=} (u(g_1 g_2), v(g_1 g_2)) \\
&= \psi(g_1 g_2)
\end{aligned}$$

thus ψ is a morphism of groups. The commutativity of the diagram is obvious. \square

The classification of unified products. In this subsection we provide the classification part of the (Gr) extending structure problem. As special cases, a general Schreier theorem for unified products is obtained and the answer to a question by Kuperberg is given.

Theorem 2.16. *Let H be a group and $(S, 1_S)$, $(S', 1_{S'})$ two pointed sets. Let $\Omega(H) = ((S, 1_S, *), \triangleleft, \triangleright, f)$ and $\Omega'(H) = ((S', 1_{S'}, *'), \triangleleft', \triangleright', f')$ be two group extending structures of H and $H \ltimes_{\Omega} S$, $H \ltimes_{\Omega'} S'$ the associated unified products. Then there exists a bijective correspondence between:*

(1) *The set of all morphisms of groups $\psi : H \ltimes_{\Omega} S \rightarrow H \ltimes_{\Omega'} S'$ such that the following diagram*

$$\begin{array}{ccc}
H & \xrightarrow{i_H} & H \ltimes_{\Omega} S \\
\downarrow \text{Id}_H & & \downarrow \psi \\
H & \xrightarrow{i_H} & H \ltimes_{\Omega'} S'
\end{array} \tag{36}$$

is commutative.

(2) The set of all pairs (r, v) , where $r : S \rightarrow H$, $v : S \rightarrow S'$ are two unitary maps such that:

$$v(s \triangleleft h) = v(s) \triangleleft' h \quad (37)$$

$$(s \triangleright h) r(s \triangleleft h) = r(s) (v(s) \triangleright' h) \quad (38)$$

$$v(s_1 * s_2) = (v(s_1) \triangleleft' r(s_2)) *' v(s_2) \quad (39)$$

$$f(s_1, s_2) r(s_1 * s_2) = r(s_1) (v(s_1) \triangleright' r(s_2)) f'(v(s_1) \triangleleft' r(s_2), v(s_2)) \quad (40)$$

for all $s, s_1, s_2 \in S$ and $h \in H$.

Through the above correspondence $\psi : H \ltimes_{\Omega} S \rightarrow H \ltimes_{\Omega'} S'$ is given by

$$\psi(h, s) = (hr(s), v(s)) \quad (41)$$

for all $h \in H$, $s \in S$. Furthermore, $\psi : H \ltimes_{\Omega} S \rightarrow H \ltimes_{\Omega'} S'$ given by (41) is an isomorphism of groups if and only if the map $v : S \rightarrow S'$ is bijective, i.e. v is an isomorphism between the right H -sets (S, \triangleleft) and (S', \triangleleft') .

Proof. A morphism of groups $\psi : H \ltimes_{\Omega} S \rightarrow H \ltimes_{\Omega'} S'$ that makes the diagram (36) commutative is uniquely determined by two maps $r = r_{\psi} : S \rightarrow H$, $v = v_{\psi} : S \rightarrow S'$ such that $\psi(1, s) = (r(s), v(s))$ for all $s \in S$. In this case ψ is given by :

$$\psi(h, s) = \psi((h, 1_S) \cdot (1_H, s)) = (h, 1_S) \cdot (r(s), v(s)) = (hr(s), v(s))$$

for all $h \in H$ and $s \in S$ (we used (24) in the last step). Now, $\psi(1_H, 1_S) = (1_H, 1_{S'})$ if and only if $r(1_S) = 1_H$ and $v(1_S) = 1_{S'}$, i.e. r and v are unitary maps. Assuming this unitary condition, we shall prove now that ψ is a morphism of groups if and only if the compatibility conditions (37) - (40) hold for the pair (r, v) . It is enough to check the condition $\psi(xy) = \psi(x)\psi(y)$ only for generators $x, y \in (H \times \{1_S\}) \cup (\{1_H\} \times S)$ of the unified product $H \ltimes_{\Omega} S$. In fact, we only need to check it for $x = (1_H, s)$, $y = (h, 1_S)$ and $x = (1_H, s_1)$, $y = (1_H, s_2)$ (in the other two cases the relation $\psi(xy) = \psi(x)\psi(y)$ is automatically fulfilled). The condition $\psi((1_H, s) \cdot (h, 1_S)) = \psi(1_H, s) \cdot \psi(h, 1_S)$ is equivalent to (37) - (38) and the condition $\psi((1_H, s_1) \cdot (1_H, s_2)) = \psi(1_H, s_1) \cdot \psi(1_H, s_2)$ is equivalent to (39) - (40).

It remains to be proved that ψ given by (41) is an isomorphism if and only if $v : S \rightarrow S'$ is a bijective map. Assume first that ψ is an isomorphism. Then v is surjective and for $s_1, s_2 \in S$ such that $v(s_1) = v(s_2)$ we have:

$$\psi(1_H, s_2) = (r(s_2), v(s_2)) = (r(s_2), v(s_1)) = \psi(r(s_2)r(s_1)^{-1}, s_1)$$

Hence $s_1 = s_2$ and v is injective. Conversely, if v is bijective then ψ is bijective, i.e. ψ is an isomorphism of groups. \square

From now on the group H and the pointed set $(S, 1_S)$ will be fixed. Let $\mathcal{GES}(H, (S, 1_S))$ be the set of all quadruples $(*, \triangleleft, \triangleright, f)$ such that $((S, 1_S, *), \triangleleft, \triangleright, f)$ is a group extending structure of H .

Definition 2.17. Two elements $(*, \triangleleft, \triangleright, f)$ and $(*', \triangleleft', \triangleright', f')$ of $\mathcal{GES}(H, (S, 1_S))$ are called *equivalent* and we denote this by $(*, \triangleleft, \triangleright, f) \sim (*', \triangleleft', \triangleright', f')$ if there exists a pair (r, v) of unitary maps $r : S \rightarrow H$, $v : S \rightarrow S$ such that v is a bijection on the set S and the compatibility conditions (37) - (40) are fulfilled.

It follows from Theorem 2.16 that $(*, \triangleleft, \triangleright, f) \sim (*', \triangleleft', \triangleright', f')$ if and only if there exists $\psi : H \rtimes_{\Omega} S \rightarrow H \rtimes_{\Omega'} S'$ an isomorphism of groups that stabilizes H , i.e. the diagram (36) is commutative. Thus, \sim is an equivalence relation on the set $\mathcal{GES}(H, (S, 1_S))$. We denote by $\mathcal{K}_{\times}^2(H, (S, 1_S))$ the quotient set $\mathcal{GES}(H, (S, 1_S)) / \sim$.

Let $\mathcal{C}(H, (S, 1_S))$ be the category whose class of objects is the set $\mathcal{GES}(H, (S, 1_S))$. A morphism $\psi : (*, \triangleleft, \triangleright, f) \rightarrow (*', \triangleleft', \triangleright', f')$ in $\mathcal{C}(H, (S, 1_S))$ is a morphism of groups $\psi : H \rtimes_{\Omega} S \rightarrow H \rtimes_{\Omega'} S$ such that the diagram (36) is commutative.

As a direct application of Theorem 2.16 we obtain the classification of the unified products in which the pointed set $\mathcal{K}_{\times}^2(H, (S, 1_S))$ plays the key role.

Corollary 2.18. (The classification theorem for unified products) *Let H be a group and $(S, 1_S)$ a pointed set. Then there exists a bijection between the set of objects of the skeleton of the category $\mathcal{C}(H, (S, 1_S))$ and $\mathcal{K}_{\times}^2(H, (S, 1_S))$.*

Remark 2.19. Results similar to Theorem 2.16 and Corollary 2.18 can be obtained in any other category where unified products (or other special cases of it) are constructed. For example, at the level of algebras the bicrossed product is known under the name of *twisted tensor product algebra*. [14, Theorem 4.4] provides, under the name of *invariance under twisting*, sufficient conditions for two bicrossed products of algebras $A \bowtie B$ and $A' \bowtie B$ to be isomorphic such that the isomorphism stabilizes B . The result in [14, Theorem 4.4] can be improved and generalized in the spirit of Theorem 2.16.

Using Theorem 2.16 we can also prove a general Schreier classification theorem for unified products such that the classical classification theorem for group extensions becomes a special case of it. For a unified product $H \rtimes_{\Omega} S$ we denote by $\pi_S : H \rtimes_{\Omega} S \rightarrow S$ the canonical projection $\pi(h, s) := s$, for all $h \in H$ and $s \in S$.

Proposition 2.20. *Let $\Omega(H) = ((S, 1_S, *), \triangleleft, \triangleright, f)$, $\Omega'(H) = ((S, 1_S, *'), \triangleleft', \triangleright', f')$ be two group extending structures of a group H and $H \rtimes_{\Omega} S$, $H \rtimes_{\Omega'} S$ the associated unified products. The following are equivalent:*

(1) *There exists a morphism $\psi : H \rtimes_{\Omega} S \rightarrow H \rtimes_{\Omega'} S$ such that the following diagram*

$$\begin{array}{ccccc} H & \xrightarrow{i_H} & H \rtimes S & \xrightarrow{\pi_S} & S \\ Id_H \downarrow & & \downarrow \psi & & \downarrow Id_S \\ H & \xrightarrow{i_H} & H \rtimes' S & \xrightarrow{\pi_S} & S \end{array} \quad (42)$$

is commutative.

(2) $\triangleleft = \triangleleft'$ and there exists a unitary map $r : S \rightarrow H$ such that $\triangleright, *$ and f are implemented by $\triangleright', *'$ and f' via r as follows:

$$s \triangleright h = r(s) (s \triangleright' h) r(s \triangleleft h)^{-1} \quad (43)$$

$$s_1 * s_2 = (s_1 \triangleleft r(s_2)) *' s_2 \quad (44)$$

$$f(s_1, s_2) = r(s_1) (s_1 \triangleright' r(s_2)) f'(s_1 \triangleleft r(s_2), s_2) r(s_1 * s_2)^{-1} \quad (45)$$

for all $s, s_1, s_2 \in S$ and $h \in H$.

Furthermore, any morphism of groups $\psi : H \ltimes S \rightarrow H \ltimes' S$ that makes the diagram (42) commutative is an isomorphism of groups and is given by

$$\psi(h, s) = (h r(s), s) \quad (46)$$

for all $h \in H, s \in S$.

Proof. Indeed, using Theorem 2.16 any morphism of groups $\psi : H \ltimes S \rightarrow H \ltimes' S$ that makes the left square of (42) commutative is given by (41) for some unique maps (u, v) . Now, such a morphism $\psi = \psi_{u,v}$ makes the right square of (42) commutative if and only if v is the identity map on S . Now the proof follows from Theorem 2.16: (37) implies that the two right actions \triangleleft and \triangleleft' should be equal while (43) – (45) are exactly (38) – (40) for $v = \text{Id}_S$ and $\triangleleft = \triangleleft'$. \square

Proposition 2.20 tells us that in order to obtain a Schreier type theorem for unified product we have to set the group H , the pointed set $(S, 1_S)$ and a right H -action \triangleleft of the group H on the set S . Let $\mathcal{SES}(H, (S, 1_S), \triangleleft)$ be the set of all triples $(*, \triangleright, f)$ such that $((S, 1_S, *), \triangleleft, \triangleright, f)$ is a group extending structure of H .

Definition 2.21. Let H be a group, $(S, 1_S)$ a pointed set and $\triangleleft : S \times H \rightarrow S$ a right action of H on S . Two elements $(*, \triangleright, f)$ and $(*', \triangleright', f')$ of $\mathcal{SES}(H, (S, 1_S), \triangleleft)$ are called *cohomologous* and we denote this by $(*, \triangleright, f) \approx (*', \triangleright', f')$ if there exists a unitary map $r : S \rightarrow H$ such that

$$s_1 * s_2 = (s_1 \triangleleft r(s_2)) *' s_2$$

$$s \triangleright h = r(s) (s \triangleright' h) r(s \triangleleft h)^{-1}$$

$$f(s_1, s_2) = r(s_1) (s_1 \triangleright' r(s_2)) f'(s_1 \triangleleft r(s_2), s_2) r(s_1 * s_2)^{-1}$$

for all $s, s_1, s_2 \in S$ and $h \in H$.

It follows from Proposition 2.20 that $(*, \triangleright, f) \approx (*', \triangleright', f')$ if and only if there exists $\psi : H \ltimes_\Omega S \rightarrow H \ltimes_{\Omega'} S$ a morphism of groups such that the diagram (42) is commutative and moreover such a morphism is an isomorphism. Thus, \approx is an equivalence relation on the set $\mathcal{SES}(H, (S, 1_S), \triangleleft)$. We denote by $\mathcal{H}_\kappa^2(H, (S, 1_S), \triangleleft)$ the quotient set $\mathcal{SES}(H, (S, 1_S), \triangleleft) / \approx$.

Let $\mathcal{D}(H, (S, 1_S), \triangleleft)$ be the category whose class of objects is the set $\mathcal{SES}(H, (S, 1_S), \triangleleft)$. A morphism $\psi : (*, \triangleright, f) \rightarrow (*', \triangleright', f')$ in $\mathcal{D}(H, (S, 1_S), \triangleleft)$ is a morphism of groups $\psi : H \ltimes_\Omega S \rightarrow H \ltimes_{\Omega'} S$ such that the diagram (42) is commutative. The category $\mathcal{D}(H, (S, 1_S), \triangleleft)$ is a groupoid, that is any morphism is an isomorphism. We obtain:

Corollary 2.22. (*The Schreier theorem for unified products*) Let H be a group, $(S, 1_S)$ a pointed set and \triangleleft a right action of H on S . Then there exists a bijection between the set of objects of the skeleton of the category $\mathcal{D}(H, (S, 1_S), \triangleleft)$ and $\mathcal{H}_{\times}^2(H, (S, 1_S), \triangleleft)$.

Thus it follows from Corollary 2.22 that $\mathcal{H}_{\times}^2(H, (S, 1_S), \triangleleft)$ is for the classification of the unified products of groups the counterpart of the second cohomology group for the classification of an extension of an abelian group by a group [20, Theorem 7.34].

As special cases of Theorem 2.16 we can derive the necessary and sufficient conditions for a unified product to be isomorphic to a crossed product respectively to a bicrossed product.

Corollary 2.23. Let H be a group, $\Omega(H) = ((S, 1_S, *), \triangleleft, \triangleright, f)$ a group extending structure of H and $(H, G, \triangleright', f')$ a crossed system of groups. The following are equivalent:

(1) There exists $\psi : H \rtimes_{\Omega} S \rightarrow H \#_{\triangleright'}^{f'} G$ an isomorphism of group such that the following diagram

$$\begin{array}{ccc} H & \xrightarrow{i_H} & H \rtimes_{\Omega} S \\ \downarrow \text{Id}_H & & \downarrow \psi \\ H & \xrightarrow{i_H} & H \#_{\triangleright'}^{f'} G \end{array} \quad (47)$$

is commutative.

(2) The following two conditions hold:

- (2a) The right action \triangleleft of $\Omega(H)$ is the trivial one, i.e. $(S, *)$ is a group structure on S and $(H, (S, *), \triangleright, f)$ is a crossed system of groups.
- (2b) There exists a pair (r, v) , where $r : S \rightarrow H$ is a unitary map, $v : (S, *) \rightarrow G$ is an isomorphism of groups such that:

$$s \triangleright h = r(s) (v(s) \triangleright' h) r(s)^{-1} \quad (48)$$

$$f(s_1, s_2) = r(s_1) (v(s_1) \triangleright' r(s_2)) f'(v(s_1), v(s_2)) r(s_1 * s_2)^{-1} \quad (49)$$

for all $s, s_1, s_2 \in S$ and $h \in H$.

Proof. We apply Theorem 2.16 in the case that \triangleleft' is the trivial action in the group extending structure $\Omega'(H) = ((G, 1_G, *'), \triangleleft', \triangleright', f')$. Using Example 2.10 in this case $(H, G, \triangleright', f')$ becomes a crossed system of groups. Then, any isomorphism ψ is uniquely determined via the formula (41) by a pair (r, v) . Moreover, as v is bijective, it follows from (37) that \triangleleft is the trivial action in the group extending structure $\Omega(H)$ while (39) shows that $v : (S, *) \rightarrow G$ is a morphism of groups, i.e. an isomorphism of groups. Finally, (38) and (40) take the forms (48) and (49). \square

Remark 2.24. Let H be a group and E be a set such that $H \subseteq E$. Corollary 2.23 classify, up to an isomorphism of groups that stabilizes the subgroup H , all groups structures \cdot that can be defined on the set E such that H is a normal subgroup of (E, \cdot) . Thus, Corollary 2.23 is a generalization of Schreier's classification theorem for extensions of an abelian group K by a group Q [20, Theorem 7.34].

Indeed, using Corollary 1.2 we can restate the problem in the equivalent language of crossed systems. Let $(H, G, \triangleright, f)$ and $(H, G, \triangleright', f')$ be two crossed systems of groups. Then there exists an isomorphism of groups $H \#_{\triangleright}^f G \cong H \#_{\triangleright'}^{f'} G$ that stabilizes H if and only if there exists a unitary map $r : G \rightarrow H$ and an automorphism $v : G \rightarrow G$ of the group G such that (48) and (49) are fulfilled.

The following Corollary gives a necessary and sufficient condition for a unified product to be isomorphic to a bicrossed product of groups such that H is stabilized. It generalizes [4, Theorem 3.3].

Corollary 2.25. *Let H be a group, $\Omega(H) = ((S, 1_S, *), \triangleleft, \triangleright, f)$ a group extending structure of H and $(H, G, \triangleleft', \triangleright')$ a matched pair of groups. The following are equivalent:*

(1) *There exists $\psi : H \ltimes_{\Omega} S \rightarrow H \bowtie G$ an isomorphism of group such that the following diagram*

$$\begin{array}{ccc} H & \xrightarrow{i_H} & H \ltimes_{\Omega} S \\ \downarrow \text{Id}_H & & \downarrow \psi \\ H & \xrightarrow{i_H} & H \bowtie G \end{array} \quad (50)$$

is commutative.

(2) *There exists a pair (r, v) , where $r : S \rightarrow H$ is a unitary map, $v : (S, \triangleleft) \rightarrow (G, \triangleleft')$ is an unitary map and an isomorphism of H -sets such that:*

$$(s \triangleright h) r(s \triangleleft h) = r(s) (v(s) \triangleright' h) \quad (51)$$

$$v(s_1 * s_2) = (v(s_1) \triangleleft' r(s_2)) v(s_2) \quad (52)$$

$$f(s_1, s_2) = r(s_1) (v(s_1) \triangleright' r(s_2)) r(s_1 * s_2)^{-1} \quad (53)$$

for all $s, s_1, s_2 \in S$ and $h \in H$.

Proof. We apply Theorem 2.16 in the case where f' is the trivial cocycle in the group extending structure $\Omega'(H) = ((G, 1_G, *), \triangleleft', \triangleright', f')$. Using Example 2.11 in this case $\Omega'(H)$ is just a matched pair of groups. Then, any isomorphism ψ is uniquely determined via the formula (41) by a pair (r, v) . Moreover, as v is bijective, it follows from (37) that v is an isomorphism of right H -sets between (S, \triangleleft) and (G, \triangleleft') . The rest is obvious. \square

Finally, as a last application of Theorem 2.16, we shall give the answer of the following question of Kuperberg [16]: let E be a group that has two exact factorizations $E = HS$ and $E = HT$ through two subgroups S and T . What is the relation between the groups S and T ?

Corollary 2.26. *Let E be a group and $H \leq E$ a subgroup of E . The following statements are equivalent:*

(1) *E has two exact factorizations $E = HS$ and $E = HT$, through two subgroups $S \leq E$ and $T \leq E$. Let $(H, S, \triangleleft, \triangleright,)$ and $(H, T, \triangleleft', \triangleright',)$ be the matched pairs of groups associated to the factorization $E = HS$ and respectively $E = HT$.*

(2) *There exists a pair (r, v) , where $r : S \rightarrow H$, $v : S \rightarrow T$ are two unitary maps such that v is bijective and the following relations hold:*

$$r(s)v(s) = s \quad (54)$$

$$v(s \triangleleft h) = v(s) \triangleleft' h \quad (55)$$

$$(s \triangleright h) r(s \triangleleft h) = r(s) (v(s) \triangleright' h) \quad (56)$$

$$v(s_1 s_2) = (v(s_1) \triangleleft' r(s_2)) v(s_2) \quad (57)$$

$$r(s_1 s_2) = r(s_1) (v(s_1) \triangleright' r(s_2)) \quad (58)$$

for all $s, s_1, s_2 \in S$ and $h \in H$.

Of course, it follows from (54) that $r(s) = sv(s)^{-1} \in H$. Thus, the second item of the above Corollary can be restated as follows: there exists a unitary bijective map $v : S \rightarrow T$ such that $sv(s)^{-1} \in H$ for all $s \in S$ and then the compatibility conditions (56) - (58) has to be rewritten with $r(s) = sv(s)^{-1}$, thus in a less transparent manner.

Proof. We translate the question to the equivalent case of bicrossed product of groups. That is we apply Takeuchi's theorem [21] in order to construct two matched pairs of groups such that the multiplication maps $u : H \bowtie S \rightarrow E$, $u(h, s) = hs$ and $u : H \bowtie' T \rightarrow E$, $u(h, t) = ht$ are isomorphisms of groups. As S and T are subgroups of E the cocycles f and f' are the trivial maps and $s * s' = a *' s' = as$, the multiplication of E . In this equivalent context, the statement (1) is equivalent to the existence of an isomorphism of groups $\psi : H \bowtie S \rightarrow H \bowtie' T$ such that the diagram

$$\begin{array}{ccccc} H & \xrightarrow{i_H} & H \bowtie S & \xrightarrow{u} & E \\ Id_H \downarrow & & \downarrow \psi & & \downarrow Id_E \\ H & \xrightarrow{i_H} & H \bowtie' T & \xrightarrow{u'} & E \end{array} \quad (59)$$

is commutative. Now, we apply Theorem 2.16: any isomorphism ψ that makes the left square of (59) commutative is given by (41) for a unique pair of unitary maps (r, v) , where v is bijective and (37) - (40) hold. The right square of (59) is commutative if and only if $r(s)v(s) = s$, for all $s \in S$, i.e. (54) holds. Finally, (55) - (58) are exactly the simplified versions of (37) - (40) corresponding to the trivial cocycles f and f' and we are done. \square

3. CONCLUSIONS AND OUTLOOK

The crossed product and the bicrossed product for groups served as models for similar constructions in other fields of study: both constructions have been already introduced at the level of algebras, Lie groups and Lie algebras, locally compact groups, Hopf algebras, locally compact quantum groups, groupoids, von Neumann algebras, etc. Thus, the construction of the unified product presented in this paper at the level of groups can be generalized to all the fields above. In fact, having the construction given in this paper as a model the unified product for quantum groups was introduced in [3] and was recently generalized [11] to the case of Hopf algebras in braided monoidal categories.

The second direction for further study is related to the existence of hidden symmetries of an H -principal bundle. Let H and G be Lie groups. To be consistent with the conventions of our construction we consider the action of H in an H -principal bundle to be a left action. The H -principal bundle (E, S, π) [15] is said to be (right) G -equivariant if there are differentiable right actions of G on E and S such that π is G -equivariant and the left action of H and the right action of G on E commute; we say that the G -action is fiber transitive if the action of G on S is transitive.

Let K be a Lie group and let (E, S, π) be an K -equivariant H -principal bundle.

Does there exist a Lie group $G \supset K$ endowed with actions on E and S that extend the corresponding K -actions such that (E, S, π) becomes G -equivariant fiber transitive?

If such a G exists we say that the bundle (E, S, π) has *hidden symmetries*. The unified product associated to an extending structure provides an answer to the above question for $E = H \times S$ (the trivial H -bundle over S), $K = H$, and the actions on S and E given respectively by $h \cdot k := s \triangleleft k$ and $(h, s) \cdot k := (h(s \triangleright k), s \triangleleft k)$, for all $h \in H$, $s \in S$, $k \in K$. If the right action of H on S has a fixed point 1_S , and S has an H -group structure then the bundle becomes $(H \ltimes_\Omega S)$ -equivariant fiber transitive.

Therefore, the existence of H -group structures on S is related to the existence of hidden symmetries. In the above discussion we suppressed any reference to the differential or algebraic structures on the H -bundle. The fact that our product construction is compatible with all the additional structures is the subject of a forthcoming study.

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